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Berry Phase for the Hamiltonian of an $so(5)$ Algebraic Structure

Hong-Biao Zhang[1,2,*] and Li-Jun Tian[3]

[1] *Institute of Theoretical Physics, Northeast Normal University, Changchun 130024, P.R. China*
[2] *Department of Mathematics, University of Queensland, Brisbane, QLD 4072, Australia*
[3] *Department of Physics, College of Science, Shanghai University, Shanghai 200436, P.R. China*

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The Hamiltonian of an $so(5)$ algebraic structure is constructed by extending an $so(3)$ system to an $so(5)$ dynamical system. We exactly solve the Hamiltonian by using the coherent state operator of the $so(5)$ algebra. It is shown that the eigenstates corresponding to the same eigenvalue are multi-fold degenerate. The Berry connection associated with the $so(5)$ coherent states is calculated and is found to be decomposed into Abelian and non-Abelian Berry connections. Based on the above theory, the Berry connections of the p-wave superconductivity model are determined explicitly.

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I. INTRODUCTION

The Berry phase has attracted much attention in quantum theory, since Berry first put forward the idea that the adiabatic phase in spin-$\frac{1}{2}$ systems has a monopole field strength in the parameter space [1]. Theoretical and experimental studies were mainly concentrated on Abelian adiabatic phases [2–4] and on those related to non-degenerate levels. On the other hand, a non-Abelian adiabatic phase, which is related to the degenerate levels of dynamical systems, was first discussed by Zee and Wilczek [5, 6]. The topological properties of the non-Albelian Berry phase for an so(2n+1) spinor were studied by Benedict *et al.* [7]. Especially, Murakami *et al.* discussed the su(2) holonomy of an $so(5)$ spinor and concluded that it is described by a Yang monopole at the degeneracy point [8–10]. Recently, the Berry phase and related geometric phases, such as those of entangled states as well as mixed states, have received renewed interest as a result of proposals for their use in the implementation of quantum computing gates [11, 12]. A coherent state is an important physical concept both theoretically and experimentally [13, 14], and it plays an important role in many quantum physical fields. Coherent states of the Landau level have been studied in [15, 16], and Berry phases for coherent states as well as squeezed coherent states of a one-dimensional harmonic oscillator have been illustrated in [17]. The Berry phase for the coherent states of Landau levels, which are highly degenerate and with an additional parameter (i.e., the magnetic field B) has been investigated [18]. These works mainly concentrate on systems related to the Heisenberg and $so(3)$ algebras. In this paper, degenerate levels of the Hamiltonian of an $so(5)$ algebraic structure are generated from an $so(3)$ system by making a simple transformation. To the best of our knowledge, the Berry phase of the Hamiltonian for an

$so(5)$ algebraic structure has never been studied in detail; it is worthy of our investigation.

This paper is organized as follows: In Sec. II we extend a Hamiltonian of $so(3)$ algebraic structure to a Hamiltonian of $so(5)$ algebraic structure. Imitating the $so(3)$ case, eigenstates and eigenvalues of the Hamiltonian are obtained through the $so(5)$ algebraic coherent operator. In Sec. III we calculate the Berry connections for the coherent states of the Hamiltonian of $so(5)$ algebraic structure. In Sec. IV, based on the above theory, the Berry connection of the p-wave superconductivity (SC) model for any momentum $\vec{k}$ is discussed. A conclusion and discussion are given in the last section.

II. THE HAMILTONIAN FOR AN $so(5)$ ALGEBRAIC STRUCTURE AND ITS DEGENERATE SOLUTIONS

First, let us briefly review the $so(5)$ algebra. Its antisymmetric generators are expressed in terms of the set $\{I_{ab}, a, b = 1, 2, \cdots, 5\}$ and satisfy the commutation relations:

$$[I_{ab}, I_{cd}] = i(\delta_{ac}I_{bd} + \delta_{bd}I_{ac} - \delta_{ad}I_{bc} - \delta_{bc}I_{ad}), \tag{1}$$

$$I_{ab} = -I_{ba}. \tag{2}$$

With the linear combination of the generators,

$$\begin{array}{l} \hat{\pi}_z^\dagger = I_{34} + iI_{42},\ \hat{\pi}_x^\dagger = I_{31} + iI_{12},\ \hat{\pi}_y^\dagger = I_{35} + iI_{52},\ \hat{Q} = I_{23},\ \ \hat{F}_x = I_{45}, \\ \hat{\pi}_z = I_{34} - iI_{42},\ \hat{\pi}_x = I_{31} - iI_{12},\ \hat{\pi}_y = I_{35} - iI_{52},\ \hat{F}_z = I_{15},\ \hat{F}_y = I_{14}, \end{array} \tag{3}$$

the commutation relations (1) are recast as

$$\begin{array}{lll} [\hat{\pi}_\alpha^\dagger, \hat{\pi}_\beta] = 2\delta_{\alpha\beta}\hat{Q} + 2i\epsilon_{\alpha\beta\gamma}\hat{F}_\gamma, & [\hat{F}_\alpha, \hat{F}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{F}_\gamma, & [\hat{F}_\alpha, \hat{\pi}_\beta] = -i\epsilon_{\alpha\beta\gamma}\hat{\pi}_\gamma, \\ [\hat{F}_\alpha, \hat{\pi}_\beta^\dagger] = -i\epsilon_{\alpha\beta\gamma}\hat{\pi}_\gamma^\dagger, & [\hat{Q}, \hat{\pi}_\alpha] = -\hat{\pi}_\alpha, & [\hat{Q}, \hat{F}_\alpha] = 0, \\ [\hat{Q}, \hat{\pi}_\alpha^\dagger] = \hat{\pi}_\alpha^\dagger, & [\hat{\pi}_\alpha, \hat{\pi}_\beta] = 0, & [\hat{\pi}_\alpha^\dagger, \hat{\pi}_\beta^\dagger] = 0. \end{array} \tag{4}$$

The quadratic Casimir operator is $\hat{C} = \hat{Q}^2 + \hat{\vec{F}}^2 + \frac{1}{2}[\hat{\vec{\pi}} \cdot \hat{\vec{\pi}}^\dagger + \hat{\vec{\pi}}^\dagger \cdot \hat{\vec{\pi}}]$. As $so(5)$ is a rank two algebra, we choose the two mutually commuting generators $\hat{Q}$ and $\hat{F}_z$ as members of the Cartan subalgebra of $so(5)$. This allows us to define the state vector $|p, q >= |p > \otimes |q >$ as their common eigenstates, i.e.,

$$\hat{Q}|p, q >= p|p, q >, \quad \hat{F}_z|p, q >= q|p, q > . \tag{5}$$

This indicates that the general irreducible representations of $so(5)$ are described by two integers or half integers, p and q. By means of the commutation relations (4), we easily see that

$$\begin{array}{l}\hat{\pi}_z^\dagger|p,q> \propto\ |p+1,q>,\\ \hat{\pi}_z|p,q> \propto\ |p-1,q>,\\ \hat{F}_\pm|p,q> \propto\ |p,q\pm 1>,\\ \hat{\pi}_\pm^\dagger|p,q> \propto\ |p+1,q\pm 1>,\\ \hat{\pi}_\pm|p,q> \propto |p-1,q\pm 1>,\end{array} \tag{6}$$

where $\hat{\pi}_\pm = \hat{\pi}_x \pm i\hat{\pi}_y$, $\hat{\pi}_\pm^\dagger = \hat{\pi}_x^\dagger \pm i\hat{\pi}_y^\dagger$, and $\hat{F}_\pm = \hat{F}_x \pm i\hat{F}_y$ are the usual shift operators.

Next, we start from an $so(3)$ system, which describes a spin-j particle interacting with an external magnetic field. Its Hamiltonian takes the form

$$\hat{H}(\vec{n}) = \vec{B}(t)\cdot\hat{\vec{Q}}, \tag{7}$$

where the operator $\hat{\vec{Q}}$ obeys the $so(3)$ algebra commutation relations, $[\hat{Q}_\alpha, \hat{Q}_\beta] = i\epsilon_{\alpha\beta\gamma}\hat{Q}_\gamma$, and $\vec{B}(t) = B\vec{n}(t)$ with the unit vector $\vec{n} = (\sin\Theta\cos\Phi, \sin\Theta\sin\Phi, \cos\Theta)$. The magnetic field $\vec{B}(t)$ acts as an external control parameter, as its direction $\vec{n}$ can be experimentally changed while its magnitude B is constant. The adiabatic phase for a spin-$\frac{1}{2}$ particle in a magnetic field was first studied by Berry [1], in which a monopole structure of the adiabatic gauge field strength was discovered. Following the general strategy, in order to diagonalize (7), one often introduces the unitary operator $\hat{U}(\vec{n})$:

$$\hat{U}(\vec{n}) = \exp\left[\frac{\Theta}{2}(e^{i\Phi}\hat{Q}_+ - e^{-i\Phi}\hat{Q}_-)\right]. \tag{8}$$

This operator, related to the coherent state of $so(3)$, is referred to as the coherent operator of $so(3)$. By direct calculation, we obtain $\hat{U}^\dagger(\vec{n})\hat{H}(\vec{n})\hat{U}(\vec{n}) = B\hat{Q}_z$. The eigenstates of $\hat{Q}_z$ are defined by $|p>$ with $p = -j, -j+1, \cdots, j-1, j$. Then the instantaneous eigenstates of $\hat{H}(\vec{n})$ are given by

$$|p(\vec{n})> = \hat{U}(\vec{n})|p> \tag{9}$$

with eigenvalue $E_p = pB$.

We now consider a complex case. $\hat{\vec{Q}}$ can be expressed in terms of a linear combination of the $so(5)$ generators:

$$\hat{Q}_+(\vec{m}) = \vec{m}\cdot\hat{\vec{\pi}}^\dagger, \quad \hat{Q}_-(\vec{m}) = \vec{m}\cdot\hat{\vec{\pi}}, \quad \hat{Q}_z = \hat{Q}; \tag{10}$$

or

$$\hat{Q}_x(\vec{m}) = \vec{m} \cdot [\frac{1}{2}(\hat{\vec{\pi}} + \hat{\vec{\pi}}^\dagger)], \quad \hat{Q}_y(\vec{m}) = \vec{m} \cdot [\frac{i}{2}(\hat{\vec{\pi}} - \hat{\vec{\pi}}^\dagger)], \quad \hat{Q}_z = \hat{Q}; \tag{11}$$

where $\vec{m}(t) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ may be regarded as an unit vector of an internal space. Substituting (10) into (7) we immediately obtain

$$\hat{H}(\vec{n}, \vec{m}) = B[\cos\Theta\hat{Q}_z + \frac{1}{2}\sin\Theta e^{i\Phi}\vec{m} \cdot \hat{\vec{\pi}} + \frac{1}{2}\sin\Theta e^{-i\Phi}\vec{m} \cdot \hat{\vec{\pi}}^\dagger]. \tag{12}$$

Although we have only made a simple transformation, the Hamiltonina (12) is significantly different from the Hamiltonian (7): not only may the Hamiltonian (12) describe an $so(5)$ dynamical system, but also it has great significance for studying p-wave superconductivity systems [19].

Substituting (10) into (8) we obtain

$$\hat{U}(\vec{n}, \vec{m}) = \exp\{\frac{\Theta}{2}[e^{i\Phi}\hat{Q}_+(\vec{m}) - e^{-i\Phi}\hat{Q}_-(\vec{m})]\} = \exp[\frac{\Theta}{2}(e^{i\Phi}\vec{m} \cdot \vec{\pi}^\dagger - e^{-i\Phi}\vec{m} \cdot \vec{\pi})], \tag{13}$$

where the unitary operator $\hat{U}(\vec{n}, \vec{m})$ associated with two different kinds of spaces is a coherent state operator of $so(5)$. Similar to the $so(3)$ case, the Hamiltonian $\hat{H}(\vec{n}, \vec{m})$ can be diagonalized by means of $\hat{U}(\vec{n}, \vec{m})$:

$$\hat{U}^\dagger(\vec{n}, \vec{m})\hat{H}(\vec{n}, \vec{m})\hat{U}(\vec{n}, \vec{m}) = B\hat{Q}_z. \tag{14}$$

Therefore, the instantaneous eigenstates of the Hamiltonian $\hat{H}(\vec{n}, \vec{m})$ with the same energy eigenvalue $E_p = pB$ are

$$|p(\vec{n}, \vec{m}), q >= \hat{U}(\vec{n}, \vec{m})|p, q >, \tag{15}$$

which are multi-fold degenerate states with different values of q, i.e., the $so(5)$ coherent states. We know that $\hat{H}(\vec{n}, \vec{m})$ has multi-fold degenerate eigenstates corresponding to the same energy eigenvalue $E_p = pB$, and the eigenstates are usually combinations of the degenerate states with the same energy:

$$|p(\vec{n}, \vec{m}) >= \sum_q f_q |p(\vec{n}, \vec{m}), q >, \tag{16}$$

where f_q are arbitrary complex numbers which make $|p(\vec{n}, \vec{m}) >$ normalized, i.e., they satisfy the condition $\sum_q |f_q|^2 = 1$.

III. BERRY PHASE FOR THE COHERENT STATES OF THE HAMILTONIAN FOR AN $so(5)$ ALGEBRAIC STRUCTURE

We can see from the preceeding section that the Hamiltonian for an $so(5)$ algebraic structure has multi-fold degenerate states with the same p, which are the $so(5)$ algebraic coherent states of this Hamiltonian. The general Berry phase for degenerate states was constructed in [6], and it may have a non-Abelian structure. In this section, we will discuss the Berry phase for the $so(5)$ algebraic coherent states case. Let us begin with the unitary operator

$$\begin{aligned}\hat{U}(\vec{n},\vec{m}) &= \exp\{\frac{\Theta}{2}[e^{i\Phi}\hat{Q}_+(\vec{m}) - e^{-i\Phi}\hat{Q}_-(\vec{m})]\} \\ &= \exp[\xi\hat{Q}_+(\vec{m})]\exp[\ln(1+|\xi|^2)\hat{Q}_z]\exp[-\xi^*\hat{Q}_-(\vec{m})] \\ &= \exp[-\xi^*\hat{Q}_-(\vec{m})]\exp[-\ln(1+|\xi|^2)\hat{Q}_z]\exp[\xi\hat{Q}_+(\vec{m})],\end{aligned}$$

where $\xi = e^{i\Phi}\tan\frac{\Theta}{2}$.

Noting $[\hat{Q}_\pm(\vec{m}), d\{\hat{Q}_\pm(\vec{m})\}] = 0$, one obtains

$$\begin{aligned}d\hat{U}(\vec{n},\vec{m}) &= \frac{1}{1+|\xi|^2}[\hat{Q}_+(\vec{m})d\xi - \hat{Q}_-(\vec{m})d\xi^* + (\xi^* d\xi - \xi d\xi^*)\hat{Q}_z]\hat{U}(\vec{n},\vec{m}) \\ &\quad +\xi d[\hat{Q}_+(\vec{m})]\hat{U}(\vec{n},\vec{m}) - \xi^*\hat{U}(\vec{n},\vec{m})d[\hat{Q}_-(\vec{m})],\end{aligned}$$

and

$$\begin{aligned}\hat{U}^\dagger(\vec{n},\vec{m})d\hat{U}(\vec{n},\vec{m}) &= \frac{1}{1+|\xi|^2}\hat{U}^\dagger(\vec{n},\vec{m})\{\hat{Q}_+(\vec{m})d\xi - \hat{Q}_-(\vec{m})d\xi^* + (\xi^* d\xi - \xi d\xi^*)\hat{Q}_z\}\hat{U}(\vec{n},\vec{m}) \\ &\quad +\xi\hat{U}^\dagger(\vec{n},\vec{m})d[\hat{Q}_+(\vec{m})]\hat{U}(\vec{n},\vec{m}) - \xi^* d[\hat{Q}_-(\vec{m})]. \qquad (17)\end{aligned}$$

In order to simplify the above relation (17), we notice the following results:

$$\begin{aligned}&\frac{1}{1+|\xi|^2}[\hat{Q}_+(\vec{m})d\xi - \hat{Q}_-(\vec{m})d\xi^* + (\xi^* d\xi - \xi d\xi^*)\hat{Q}_z] \qquad (18) \\ &= \frac{e^{i\Phi}}{2}(d\Theta + i\sin\Theta d\Phi)\hat{Q}_+(\vec{m}) - \frac{e^{-i\Phi}}{2}(d\Theta - i\sin\Theta d\Phi)\hat{Q}_-(\vec{m}) + 2i\sin^2(\frac{\Theta}{2})\hat{Q}_z d\Phi,\end{aligned}$$

and

$$\begin{aligned}\hat{U}^\dagger(\vec{n},\vec{m})[d\hat{Q}_+(\vec{m})]\hat{U}(\vec{n},\vec{m}) &= \cos^2(\frac{\Theta}{2})d[\hat{Q}_+(\vec{m})] + \sin^2(\frac{\Theta}{2})e^{-i2\Phi}d[\hat{Q}_-(\vec{m})] \\ &\quad +i[(2\sin\phi d\theta + \frac{1}{2}\sin 2\theta\cos\phi d\phi)\hat{F}_x + \sin^2\theta d\phi\hat{F}_z \\ &\quad -(2\cos\phi d\theta - \frac{1}{2}\sin 2\theta\sin\phi d\phi)\hat{F}_y]\sin\Theta e^{-i\Phi}. \qquad (19)\end{aligned}$$

Substituting (18) and (19) into (17) one gets

$$\begin{aligned}\hat{U}^{\dagger}(\vec{n},\vec{m})d\hat{U}(\vec{n},\vec{m}) &= \frac{1}{2}[(d\Theta+i\sin\Theta d\Phi)e^{i\Phi}\hat{Q}_{+}(\vec{m})-(d\Theta-i\sin\Theta d\Phi)e^{-i\Phi}\hat{Q}_{-}(\vec{m})]\\ &+2id\Phi\sin^{2}(\frac{\Theta}{2})\hat{Q}_{z}+i\sin^{2}(\frac{\Theta}{2})[(4\sin\phi d\theta+\sin 2\theta\cos\phi d\phi)\hat{F}_{x}\\ &+2d\phi\sin^{2}\theta\hat{F}_{z}-(4\cos\phi d\theta-\sin 2\theta\sin\phi d\phi)\hat{F}_{y}]\\ &+\frac{1}{2}\sin\Theta[e^{i\Phi}d\hat{Q}_{+}(\vec{m})-e^{-i\Phi}d\hat{Q}_{-}(\vec{m})]. \qquad (20)\end{aligned}$$

Then we can calculate the Berry connection in the degenerate space as follows:

$$\begin{aligned}A^{q'q} &= <p(\vec{n},\vec{m}),q'|d|p(\vec{n},\vec{m}),q>=<p,q'|\hat{U}^{\dagger}d\hat{U}|p,q>\\ &= 2ip\sin^{2}(\frac{\Theta}{2})\delta_{qq'}d\Phi+i\sin^{2}(\frac{\Theta}{2})[2q\sin^{2}\theta\delta_{q'q}d\phi\\ &\quad+(4\sin\phi d\theta+\sin 2\theta\cos\phi d\phi)<q',p|\hat{F}_{x}|p,q>\\ &\quad-(4\cos\phi d\theta-\sin 2\theta\sin\phi d\phi)<q',p|\hat{F}_{y}|p,q>]\\ &= A_{\Theta}d\Theta+A_{\Phi}d\Phi+A_{\theta}d\theta+A_{\phi}d\phi. \qquad (21)\end{aligned}$$

The components of the Berry connection A are

$$A_{\Theta}^{q'q}=0,\quad A_{\Phi}^{q'q}=2ip\sin^{2}(\frac{\Theta}{2})\delta_{qq'}=ip(1-\cos\Theta)\delta_{qq'}, \qquad (22)$$

and

$$\begin{aligned}A_{\theta}^{q'q} &= 4i\sin^{2}(\frac{\Theta}{2})[\sin\phi<p,q'|\hat{F}_{x}|p,q>-\cos\phi<p,q'|\hat{F}_{y}|p,q>], \qquad (23)\\ A_{\phi}^{q'q} &= i\sin^{2}(\frac{\Theta}{2})[2q\sin^{2}\theta\delta_{q'q}+\sin 2\theta(\cos\phi<p,q'|\hat{F}_{x}|p,q>+\sin\phi<p,q'|\hat{F}_{y}|p,q>)].\end{aligned}$$

From (6), we have $\hat{F}_{\pm}|p,q>=f_{\pm}(p,q)|p,q\pm 1>$, where $f_{\pm}(p,q)$ is the proportionality coefficient between $|p,q>$ and $|p,q\pm 1>$. Then the relation (23) can expressed as

$$\begin{aligned}A_{\theta}^{q'q} &= 2\sin^{2}(\frac{\Theta}{2})[e^{i\phi}f_{-}(p,q)\delta_{q'q-1}-e^{-i\phi}f_{+}(p,q)\delta_{q'q+1}], \qquad (24)\\ A_{\phi}^{q'q} &= i\sin^{2}(\frac{\Theta}{2})\{2q\sin^{2}\theta\delta_{q'q}+\frac{1}{2}\sin 2\theta[e^{-i\phi}f_{+}(p,q)\delta_{q'q+1}+e^{i\phi}f_{-}(p,q)\delta_{q'q-1}]\}.\end{aligned}$$

We note that A_{Θ} and A_{Φ} in (22) are Abelian. With the help of the adiabatic theorem for degenerate states proved in [20, 21], one knows that $|f_q|$ in (16) will not change during the arbitrary slow evolution of $\vec{n}$ and $\vec{m}$. In addition, because the Berry connections of Θ and

Φ are Abelian, the f_q will provide a Berry phase factor which is the same for all q after an adiabatic evolution in external space. The Berry phase is

$$\gamma_p = i\int_0^{2\pi} A_\Phi d\Phi = -2\pi p(1-\cos\Theta), \tag{25}$$

which is the same as that for the non-degenerate coherent states of $so(3)$. This Berry phase comes from the slow evolution of the external magnetic field.

Moreover, we see from (24) that $A_\theta^{q'q}$ and $A_\phi^{q'q}$ are non-Abelian. They are generated from the slow evolution of the internal space freedom, and will provide a non-Abelian Berry phase. Therefore, the connection for such an $so(5)$ coherent state system can naturally be decomposed into an Abelian and non-Abelian Berry phase.

IV. BERRY PHASE FOR A P-WAVE SUPERCONDUCTIVITY PHYSICAL MODEL

In this section we will apply the above theory to a specific physical model, i.e., the p-wave superconductivity (SC) model. The Hamiltonian takes the Anderson reduced form [22, 23]

$$\hat{H}_{SC} = \sum_{\mathbf{k}\alpha} \epsilon_{\mathbf{k}} \hat{a}^\dagger_{\mathbf{k}\alpha}\hat{a}_{\mathbf{k}\alpha} + \sum_{\mathbf{k},\mathbf{k}',\alpha,\beta} V_{\mathbf{k}\mathbf{k}'}\hat{a}^\dagger_{\mathbf{k}'\alpha}\hat{a}^\dagger_{-\mathbf{k}'\beta}\hat{a}_{-\mathbf{k}\beta}\hat{a}_{\mathbf{k}\alpha}, \tag{26}$$

where $\epsilon_{\mathbf{k}} = \frac{k^2}{2m} - \nu$, $\alpha, \beta = \uparrow, \downarrow$, and for the p-wave SC, $V_{\mathbf{k}\mathbf{k}'} = -3V_1(k,k')\hat{\mathbf{k}}\cdot\hat{\mathbf{k}}'$ ($\hat{\mathbf{k}} = \frac{\mathbf{k}}{k}$). We introduce a realization of $so(5)$:

$$\begin{cases} \hat{\vec{\pi}}_{\mathbf{k}} = \hat{a}_{-\mathbf{k}\alpha}(\sigma_y\vec{\sigma})_{\alpha\beta}\hat{a}_{\mathbf{k}\beta}, \\ \hat{\vec{F}}_{\mathbf{k}} = \frac{1}{2}(\hat{a}^\dagger_{\mathbf{k}\alpha}\vec{\sigma}_{\alpha\beta}\hat{a}_{\mathbf{k}\beta} + \hat{a}^\dagger_{-\mathbf{k}\alpha}\vec{\sigma}_{\alpha\beta}\hat{a}_{-\mathbf{k}\beta}), \\ \hat{Q}_{\mathbf{k}} = \frac{1}{2}[\sum\limits_{\alpha=\uparrow,\downarrow}(\hat{a}^\dagger_{\mathbf{k}\alpha}\hat{a}_{\mathbf{k}\alpha} + \hat{a}^\dagger_{-\mathbf{k}\alpha}\hat{a}_{-\mathbf{k}\alpha}) - 2]. \end{cases} \tag{27}$$

It can be checked that $\hat{\vec{\pi}}_{\mathbf{k}}, \hat{\vec{F}}_{\mathbf{k}}$, and $\hat{Q}_{\mathbf{k}}$ defined above satisfy the commutation relations (4). So the Hamiltonian (26) can be rewritten in terms of the generators of $so(5)$ as

$$\hat{H}_{SC} = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}}(\hat{Q}_{\mathbf{k}} + 1) + \frac{1}{2}\sum_{\mathbf{k}\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'}\hat{\vec{\pi}}^\dagger_{\mathbf{k}'}\cdot\hat{\vec{\pi}}_{\mathbf{k}}. \tag{28}$$

Under the mean-field approximation, the Hamiltonian (28) can be linearized with respect to the $so(5)$ generators:

$$\hat{H}_{mf} = \sum_{\mathbf{k}} [\hat{H}_{\mathbf{k}} - E_*(\mathbf{k})], \tag{29}$$

with

$$\hat{H}_{\mathbf{k}} = \epsilon_{\mathbf{k}} \hat{Q}_{\mathbf{k}} + \vec{\Delta}_{\mathbf{k}} \cdot \hat{\vec{\pi}}^{\dagger}_{\mathbf{k}} + \vec{\Delta}^{*}_{\mathbf{k}} \cdot \hat{\vec{\pi}}_{\mathbf{k}}, \tag{30}$$

$$E_*(\mathbf{k}) = \epsilon_{\mathbf{k}} - \tilde{\mathbf{\Delta}}_{\mathbf{k}} \cdot < \hat{\vec{\pi}}^{\dagger}_{\mathbf{k}} >, \tag{31}$$

where $\vec{\Delta}_{\mathbf{k}} = \frac{1}{2} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} < \hat{\vec{\pi}}_{\mathbf{k}} >$ is the superconducting gap, and $< \cdots >$ is the average over both quantum states and thermodynamics. Note that the Hamiltonian $\hat{H}_{\mathbf{k}}$ for any $\mathbf{k}$ is the same as (12). Then $\hat{U}^{\dagger}(\vec{n}_{\mathbf{k}}, \vec{m}_k) \hat{H}_{\mathbf{k}} \hat{U}(\vec{n}_{\mathbf{k}}, \vec{m}_{\mathbf{k}})$ becomes diagonal for any $\mathbf{k}$:

$$\hat{U}^{\dagger}(\vec{n}_{\mathbf{k}}, \vec{m}_k) \hat{H}_{\mathbf{k}} \hat{U}(\vec{n}_{\mathbf{k}}, \vec{m}_{\mathbf{k}}) = \sqrt{\epsilon^2_{\mathbf{k}} + |\vec{\Delta}_{\mathbf{k}}|^2} \hat{Q}_{\mathbf{k}}, \tag{32}$$

where

$$\tan\Theta = \frac{|\vec{\Delta}_{\mathbf{k}}|}{\epsilon_{\mathbf{k}}}, \;\; \vec{\Delta_{\mathbf{k}}} = \frac{1}{2} \sum_{\mathbf{k}'} V_{\mathbf{k}\mathbf{k}'} < \hat{\vec{\pi}}_{\mathbf{k}} >= |\vec{\Delta}_{\mathbf{k}}| e^{i\Phi} \vec{m}_{\mathbf{k}}. \tag{33}$$

The eigenstates and eigenvalues in the particle number picture are given by

$$|p(\vec{n}_{\mathbf{k}}, \vec{m}_{\mathbf{k}}), q >= \hat{U}(\vec{n}_{\mathbf{k}}, \vec{m}_{\mathbf{k}}) |p, q >, \; E_{\mathbf{k}} = p\sqrt{\epsilon^2_{\mathbf{k}} + |\vec{\Delta}_{\mathbf{k}}|^2}, \tag{34}$$

with $|p, q >= |(n_{\mathbf{k}\alpha}), (n_{-\mathbf{k}\alpha}) >$ including 16 different kinds of states, the quantum numbers $p = \frac{1}{2}[\sum_{\alpha=\uparrow,\downarrow} (n_{\mathbf{k}\alpha} + n_{-\mathbf{k}\alpha}) - 2]$, and $q = \frac{1}{2}(n_{\mathbf{k}\uparrow} + n_{-\mathbf{k}\uparrow} - n_{\mathbf{k}\downarrow} - n_{-\mathbf{k}\downarrow})$.

Let us distinguish four cases corresponding to four different energy values.

(i). $p = \pm 1, q = 0$, then $|p = \pm 1, q = 0 >$ is written as

$$\begin{cases} |p = -1, q = 0 >= |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 0 >, \\ |p = 1, q = 0 >= |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 1 > . \end{cases} \tag{35}$$

They correspond to the ground state and the highest excited state, respectively. From Eq. (22) and Eq. (24),

$$A_\theta = A_\phi = A_\Theta = 0, \; A_\Phi = \mp 2i \sin^2 (\frac{\Theta}{2}). \tag{36}$$

It follows that there exists a $U(1)$ Abelian connection in the ground state and the highest state. The Berry connection 1-forms for the ground state and the highest state are $A = \mp 2i\sin^2(\frac{\Theta}{2})d\Phi$ and the curvature 2-forms are

$$F = dA = \mp i \sin\Theta d\Theta \wedge d\Phi. \tag{37}$$

The first Chern numbers are $\frac{i}{2\pi}\oint_{S^2}\vec{F}\cdot d\vec{S} = \pm 2$.

(ii). $|p = -\frac{1}{2}, q = \pm\frac{1}{2} >$ includes four states divided into two different types:

$$\begin{cases} |p = -\frac{1}{2}, q = \frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 0 > \\ |p = -\frac{1}{2}, q = -\frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 0 >, \end{cases} \tag{38}$$

and

$$\begin{cases} |p = -\frac{1}{2}, q = \frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 0 > \\ |p = -\frac{1}{2}, q = -\frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 1 > . \end{cases} \tag{39}$$

They correspond to the first excited state, $\hat{U}(\vec{n}_{\mathbf{k}}, \vec{m}_{\mathbf{k}})|p = -\frac{1}{2}, q = \pm\frac{1}{2} >$, and the first excited energy value is $E_{\mathbf{k}} = -\frac{1}{2}\sqrt{\epsilon_{\mathbf{k}}^2 + |\vec{\Delta}_{\mathbf{k}}|^2}$. Moreover, Eq. (22) and Eq. (24) read

$$A_\Theta = 0,\ A_\Phi = -i\sin^2(\frac{\Theta}{2})\mathbb{1}, \tag{40}$$

and

$$A_\theta = 2i\sin^2(\frac{\Theta}{2})(\sin\phi\sigma_x - \cos\phi\sigma_y),\ A_\phi = i\sin^2(\frac{\Theta}{2})[\sin^2\theta\sigma_z + \frac{1}{2}\sin 2\theta(\cos\phi\sigma_x + \sin\phi\sigma_y)]. \tag{41}$$

Therefore the connection 1-form is

$$A = i\sin^2(\frac{\Theta}{2})\{2(\sin\phi\sigma_x - \cos\phi\sigma_y)d\theta + [\sin^2\theta\sigma_z + \frac{1}{2}\sin 2\theta(\cos\phi\sigma_x + \sin\phi\sigma_y)]d\phi - \mathbb{1}d\Phi\}. \tag{42}$$

Here we note that the connection 1-form can be decomposed into a $U(1)$ Abelian connection and an $SU(2)$ non-Abelian connection. This indicates that an Abelian connection and a non-Abelian connection coexist in the first excited state. The components of the curvature

2-form $F = dA + A\wedge A$ are

$$
\begin{aligned}
F_{\Theta\Phi} &= -\frac{i}{2}\sin\Theta \mathbb{1},\\
F_{\Theta\phi} &= \frac{i}{2}\sin\Theta[\sin^2\theta\sigma_z + \frac{1}{2}\sin 2\theta(\cos\phi\sigma_x + \sin\phi\sigma_y)],\\
F_{\Theta\theta} &= i\sin\Theta(\sin\phi\sigma_x - \cos\phi\sigma_y),\\
F_{\theta\phi} &= i\sin^2(\frac{\Theta}{2})[\cos\Theta\sin 2\theta\sigma_z - (1 + 2\cos\Theta\sin^2\theta)(\cos\phi\sigma_x + \sin\phi\sigma_y)],\\
F_{\Phi\theta} &= F_{\Phi\phi} = 0.
\end{aligned}
$$

Therefore the first Chern class is given by $c_1 = \frac{i}{2\pi}TrF = -i\sin\Theta d\Theta\wedge d\Phi$, and the first Chern number is

$$c_1 = \frac{i}{2\pi}\oint_{S^2} TrF\cdot d\vec{S} = 2. \tag{43}$$

(iii). When $p = 0, q = 0, \pm 1$, then $|p, q>$ is expressed in detail as

$$|p = 0, q = 1 >= |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 0 >\equiv |(\uparrow), (\uparrow) >, \tag{44}$$

$$|p = 0, q = -1 >= |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 1 >\equiv |(\downarrow), (\downarrow) >, \tag{45}$$

$$|p = 0, q = 0 >= \begin{cases} |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 0 >= |(\uparrow\downarrow), (00) >,\\ |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 1 >= |(\uparrow), (\downarrow) >,\\ |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 0 >= |(\downarrow), (\uparrow) >,\\ |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 1 >= |(00), (\uparrow\downarrow) > . \end{cases} \tag{46}$$

We may divide them into triplet p-wave pairing states

$$|(\uparrow), (\downarrow) >, \quad \frac{1}{\sqrt{2}}[|(\uparrow), (\downarrow) > +|(\downarrow), (\uparrow) >], \quad |(\downarrow), (\downarrow) >, \tag{47}$$

and three singlet states

$$|(\uparrow\downarrow), (00) >, \quad \frac{1}{\sqrt{2}}[|(\uparrow), (\downarrow) > -|(\downarrow), (\uparrow) >], \quad |(00), (\uparrow\downarrow) > . \tag{48}$$

For the p-wave pairing states, Eq. (22) and Eq. (24) become

$$A_\Theta = A_\Phi = 0, \tag{49}$$

$$A_\theta = 4i\sin^2(\frac{\Theta}{2})(\sin\phi F_x - \cos\phi F_y), \tag{50}$$

$$A_\phi = i\sin^2(\frac{\Theta}{2})[2\sin^2\theta F_z + \sin 2\theta(\cos\phi F_x + \sin\phi F_y)], \tag{51}$$

where $F_x = \frac{1}{\sqrt{2}}\begin{pmatrix}0&1&0\\1&0&1\\0&1&0\end{pmatrix}$, $F_y = \frac{1}{\sqrt{2}}\begin{pmatrix}0&-i&0\\i&0&-i\\0&i&0\end{pmatrix}$, $F_z = \begin{pmatrix}1&0&0\\0&0&0\\0&0&1\end{pmatrix}$. Therefore, the connection 1-form is

$$A = i\sin^2(\frac{\Theta}{2})\{4(\sin\phi F_x - \cos\phi F_y)d\theta + [2\sin^2\theta F_z + \sin 2\theta(\cos\phi F_x + \sin\phi F_y)]d\phi\}. \quad (52)$$

This connection only includes a non-Abelian $SU(2)$ connection, which comes from anisotropy of the p-wave superconducting pair gap. By a simple calculation, we obtain the components of the curvature 2-form:

$$\begin{aligned} F_{\Theta\theta} &= 2i\sin\Theta(\sin\phi F_x - \cos\phi F_y), \\ F_{\Theta\phi} &= \frac{i}{2}\sin\Theta[2\sin^2\theta F_z + \sin 2\theta(\cos\phi F_x + \sin\phi F_y)], \\ F_{\theta\phi} &= i\sin^2(\frac{\Theta}{2})[(2\cos\Theta - 1)\sin 2\theta F_z \\ &\quad +(4\cos\Theta\cos 2\theta - 4\cos\Theta - 2\cos 2\theta)(\cos\phi F_x + \sin\phi F_y)]. \end{aligned}$$

This shows that the first Chern class is $c_1 = \frac{i}{2\pi}TrF = 0$, and the Chern number is given by $c_1 = \frac{i}{2\pi}\oint_{S^2} TrF\cdot d\vec{S} = 0$. Thus the Berry phase does not exist in the second excited energy for the p-wave superconducting pair.

(iv). $p = \frac{1}{2}, q = \pm\frac{1}{2}$. Similar to case (ii), the state $|p = \frac{1}{2}, q = \pm\frac{1}{2} >$ includes four states divided into two different groups, i.e.,

$$\left\{ \begin{array}{l} |p = \frac{1}{2}, q = \frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 0 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 1 >, \\ |p = \frac{1}{2}, q = -\frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 0, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 1 >, \end{array} \right. \quad (53)$$

and

$$\left\{ \begin{array}{l} |p = \frac{1}{2}, q = \frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 1, n_{-\mathbf{k}\downarrow} = 0 >, \\ |p = \frac{1}{2}, q = -\frac{1}{2} >= |n_{\mathbf{k}\uparrow} = 1, n_{\mathbf{k}\downarrow} = 1 > \otimes |n_{-\mathbf{k}\uparrow} = 0, n_{-\mathbf{k}\downarrow} = 1 > . \end{array} \right. \quad (54)$$

Eq. (22) and Eq. (24) give rise to

$$A_\Theta = 0, \quad A_\Phi = i\sin^2(\frac{\Theta}{2})\mathbb{1}, \quad (55)$$

and

$$A_\theta = 2i\sin^2(\frac{\Theta}{2})(\sin\phi\sigma_x - \cos\phi\sigma_y),\ A_\phi = i\sin^2(\frac{\Theta}{2})[\sin^2\theta\sigma_z + \frac{1}{2}\sin 2\theta(\cos\phi\sigma_x + \sin\phi\sigma_y)].$$

(56)

Thus the connection 1-form is

$$A = i\sin^2(\frac{\Theta}{2})\{2(\sin\phi\sigma_x - \cos\phi\sigma_y)d\theta + [\sin^2\theta\sigma_z + \frac{1}{2}\sin 2\theta(\cos\phi\sigma_x + \sin\phi\sigma_y)]d\phi + \mathbb{1}d\Phi\}. \quad (57)$$

It is seen that the connection 1-form can be decomposed into a $U(1)$ Abelian connection and an $so(3)$ non-Abelian connection. This indicates that an Abelian connection and a non-Abelian connection can coexist in these excited states. The components of the curvature 2-form $F = dA + A\wedge A$ are

$$\begin{aligned}
F_{\Theta\Phi} &= \frac{i}{2}\sin\Theta\mathbb{1}, \\
F_{\Theta\phi} &= \frac{i}{2}\sin\Theta[\sin^2\theta\sigma_z + \frac{1}{2}\sin 2\theta(\cos\phi\sigma_x + \sin\phi\sigma_y)], \\
F_{\Theta\theta} &= i\sin\Theta(\sin\phi\sigma_x - \cos\phi\sigma_y), \\
F_{\theta\phi} &= i\sin^2(\frac{\Theta}{2})[\cos\Theta\sin 2\theta\sigma_z - (1 + 2\cos\Theta\sin^2\theta)(\cos\phi\sigma_x + \sin\phi\sigma_y)], \\
F_{\Phi\theta} &= F_{\Phi\phi} = 0.
\end{aligned}$$

Therefore the first Chern class is $c_1 = \frac{i}{2\pi}TrF = i\sin\Theta d\Theta\wedge d\Phi$, and the first Chern number is

$$c_1 = \frac{i}{2\pi}\oint_{S^2} TrF\cdot d\vec{S} = -2. \quad (58)$$

V. CONCLUSION

In this paper, we have extended an $so(3)$ system to an $so(5)$ system and obtained the Hamiltonian for an $so(5)$ algebraic structure. The eigenstates and eigenvalues of the Hamiltonian can be given by using the coherent operator of the $so(5)$ algebra. We have calculated the Berry connections for the $so(5)$ algebraic coherent states of our $so(5)$ system, and shown that it consists of an Abelian Berry connection and a non-Abelian Berry connection. The connection of the p-wave superconductivity model in the $so(5)$ algebraic structure frame is discussed in detail.

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References

* Electronic address: `zhanghb017@nenu.edu.cn`
[1] M. V. Berry, Proc. R. Soc. Lond. A **392**, 54 (1984).
[2] S. Barry, Phys. Rev. Lett. **51**, 2167 (1983).
[3] J. Schlienz and G. Mahler, Phys. Rev. A **52**, 4396 (1995).
[4] I. L. Chuang and Y. Yamamoto, Phys. Rev. A **52**, 3489 (1995).
[5] Geometric Phase in Physics, eds. A. Shapere and F. Wilczek (World Scientific, Singapore 1989).
[6] F. Wilczek and A. Zee, Phys. Rev. Lett. **52**, 2111 (1999).
[7] M. G. Benedict *et al.*, J. Math. Phys. **30**, 1727 (1989).
[8] S. Murakami, N. Nagaosa, and S. C. Zhang, Science **301**, 1348 (2003).
[9] S. Murakami, N. Nagaosa, and S. C. Zhang, Phys. Rev. B **69**, 235206 (2004).
[10] E. Demlber and S. C. Zhang, Ann. Phys., NY **271**, 83 (1999).
[11] E. Sjöqvist, Phys. Rev. A **62**, 022109 (2000).
[12] E. Sjöqvist, A. K. Pati, A. Ekert *et al.*, Phys. Rev. Lett. **85**, 2845 (2000).
[13] R. J. Glauber, Phys. Rev. Lett. **10**, 84 (1963).
[14] W. M. Zhang, D. H. Feng and R. Gilmore, Rev. Mod. Phys. **62**, 867 (1990).
[15] M. K. Fung and Y. F. Wang, Chin. J. Phys. **37**, 435 (1999).
[16] H. Fakhri, J. Phys. A: Math. and Gen. **37**, 5203 (2004).
[17] S. Chaturvedi, M. S. Sriram, and V. Srinivasan, J. Phys. A: Math. and Gen. **20**, L1091 (1987).
[18] W. L. Yang and J. L. Chen, Phys. Rev. A **75**, 024101 (2007).
[19] H. B. Zhang, M. L. Ge, and K. Xue, J. Phys. A: Math. and Gen. **35**, L7-L11. (2002).
[20] T. Kato, J. Phys. Soc. Jpn. **5**, 435 (1950).
[21] J. E. Avron, R. Seiler, and L. G. Yaffe, Commun. Math. Phys. **110**, 33 (1987).
[22] P. W. Anderson and W.F. Brinkman, *Theory of Anisotropic Superfluidity in* ^3H_e Part II, eds. K. H. Bennemann and J. B. Ketterson (Wiley, New York, 1987), Chap. 3.
[23] R. Balian and N.R. Werthamer, Phy. Rev. **131**, 1553 (1963).